# Equilibrium Temperature of Small Body in Shearing Gas Flow

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**Abstract.** A convex body, with high thermal conductivity, is immersed in a nonuniformly flowing gas. The body is small compared to the mean free path, which in turn is small compared to the macroscopic length scale of the gas. The equilibrium temperature $T_w$ of the body is calculated. For an axially symmetric body in a simply shearing gas of temperature $T$ one obtains the equilibrium temperature

$$\frac{T_w}{T} = 1 + \frac{\beta a}{8} \frac{p_{xy}}{p} \sin^2 \theta \sin(2\varphi).$$

(this is for the case that the body is at rest with respect to the gas). $\theta, \varphi$ are polar angles of the axis of the body ($z$ is the polar axis). $a$ is a geometric shape factor of the body (which vanishes for a sphere) and $\beta$ depends on the Sonine coefficients. $\beta$ takes the value 1 if only the lowest order Sonine term is retained. $p$ is the pressure and $p_{xy}$ (the non-vanishing component of) the viscous pressure tensor.

## INTRODUCTION

The phenomenon of thermophoresis - the force acting on a small body in a gas with a temperature gradient - is in the limit of freee molecular flow caused by the non-Maxwellian nature of the molecular distribution function, see [1]. There is an analogous phenomenon: in a shearing gas, due to the non-Maxwellian character of the molecular distribution function, there will be a heat transfer to a small body. - For a general review of free molecular flow, see Schaaf [2]. See also Sone [3].

Bell and Schaaf [4], see also Schaaf [2] calculated the heat transfer to an infinite circular cylinder in the free molecular flow approximation. They used the first order Chapman-Enskog distribution function. As a result they found a contribution to the heat transfer from the shearing of the gas. Consequently, the equilibrium temperature of the body was also shown to be affected by the shearing of the gas.

In the present work, we consider a convex body, which is small compared to the mean free path. The contribution from the mean flow of the gas is calculated up to second order in the Mach number and the contribution from the shearing is calculated to first order in the Mach number and first order in the Knudsen number. We assume free molecular flow, see Kogan [5]. We also use the common, test body, approximation of neglecting the influence of the body on the incoming flow of gas molecules.

We use the Maxwell model for the surface assuming that a fraction $\alpha_\tau$ of the incoming particles is diffusively reflected, whereas the fraction $(1 - \alpha_\tau)$ is specularly reflected. We further assume that the surface of the body is characterized by the energy accomodation coefficient $\alpha_e$. See Kogan [5]. The gas is taken to be monatomic.

The equilibrium temperature is first derived for a general distribution function. This is first applied to a homogeneous flow and then to a shearing flow with zero velocity at the body. The case of a shearing flow with non-zero velocity at the body is in the approximation used given by superposition of the two contributions.

## GENERAL DISTRIBUTION FUNCTION

### Net influx of kinetic energy

Let us now start by writing down some relations of general validity. See Kogan [5].

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**14. ABSTRACT**
A convex body, with high thermal conductivity, is immersed in a nonuniformly flowing gas. The body is small compared to the mean free path, which in turn is small compared to the macroscopic length scale of the gas. The equilibrium temperature Tw of the body is calculated. For an axially symmetric body in a simply shearing gas of temperature T one obtains the equilibrium temperature (this is for the case that the body is at rest with respect to the gas). 0, (p are polar angles of the axis of the body (z is the polar axis), a is a geometric shape factor of the body (which vanishes for a sphere) and /3 depends on the Sonine coefficients. /3 takes the value 1 if only the lowest order Sonine term is retained, p is the pressure and pxy (the non- vanishing component of) the viscous pressure tensor.

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The Maxwellian distribution is written

$$f^{(0)} = n \left( \frac{m}{2\pi k_B T} \right)^{3/2} e^{-\mathcal{C}^2} . \tag{1}$$

Here, $n$ is the number density of molecules, $k_B$ the Boltzmann constant, $T$ the temperature, and $m$ the molecular mass. $\mathcal{C}_i$ is the dimensionless velocity

$$\mathcal{C}_i = \sqrt{\frac{m}{2k_B T}} C_i, \tag{2}$$

where $C_i$ is the ordinary molecular velocity.

For a general distribution, the total influx of particles is

$$N_{(i)} = -\int_{\mathbf{n}\cdot\mathbf{C}<0} (\mathbf{C}\cdot\mathbf{n}) f \mathrm{d}^3 C. \tag{3}$$

In particular, for a Maxwellian distribution $N_{(i)}$ is given by

$$N_M = n\sqrt{\frac{k_B T}{2\pi m}}. \tag{4}$$

The conservation of particles at the surface is expressed by

$$N_{(i)} = n_r\sqrt{\frac{k_B T_r}{2\pi m}} = n_w\sqrt{\frac{k_B T_w}{2\pi m}}. \tag{5}$$

Index $r$ refers to reflected and $w$ to wall.

The total influx of kinetic energy is

$$E_{(i)} = -\frac{m}{2}\int_{\mathbf{n}\cdot\mathbf{C}<0} C^2 (\mathbf{C}\cdot\mathbf{n}) f \mathrm{d}^3 C. \tag{6}$$

In particular, for a Maxwellian it takes the value

$$E_M = \frac{mn\pi}{2} \left( \frac{2k_B T}{\pi m} \right)^{3/2}. \tag{7}$$

According to the Maxwell model, the outflux of particles is given by

$$E_{(r)} = (1 - \alpha_\tau) E_{(i)} + \alpha_\tau E_M(T_r). \tag{8}$$

Thus the net influx of kinetic energy is

$$E = E_{(i)} - E_{(r)} = \alpha_\tau (E_{(i)} - E_M(T_r)). \tag{9}$$

Let us also introduce the correspondence of (8) for wall conditions

$$E_{(w)} = (1 - \alpha_\tau) E_{(i)} + \alpha_\tau E_M(T_w). \tag{10}$$

The energy accomodation coefficient is given by

$$\alpha_e = \frac{E_{(i)} - E_{(r)}}{E_{(i)} - E_{(w)}}. \tag{11}$$

Using (5) we find the net influx of kinetic energy as

$$E = \alpha_e \alpha_\tau (E_{(i)} - E_M(T_w)) \tag{12}$$

$$= \alpha_e \alpha_\tau \left( E_{(i)} - \frac{T_w}{T}\frac{n_r T_r^{1/2}}{n T^{1/2}} E_M \right) = \alpha_e \alpha_\tau E_M \left( \frac{E_{(i)}}{E_M} - \frac{T_w}{T}\frac{N_{(i)}}{N_M} \right)$$

## Equilibrium temperature for high conductivity body

Let us now for simplicity assume that $\alpha_e$ and $\alpha_\tau$ are constant. We further assume that the thermal conductivity of the body is so high, that $T_w$ is uniform in the body. The total influx of kinetic energy to the body can then be written ($S$ is the area of the body)

$$\int E \, ds = \alpha_e \alpha_\tau E_M S [(\overline{\frac{E_{(i)}}{E_M}}) - \frac{T_w}{T}(\overline{\frac{N_{(i)}}{N_M}})]. \tag{13}$$

The overbar denotes surface average

$$\overline{f} = \frac{1}{S} \int f \, ds.$$

At stationary conditions, the (equilibrium) temperature of the body is thus

$$\frac{T_w}{T} = (\overline{\frac{E_{(i)}}{E_M}}) / (\overline{\frac{N_{(i)}}{N_M}}). \tag{14}$$

# HOMOGENEOUS FLOW

Let us now consider a homogeneous flow of the gas. See Bird [8]. We introduce the dimensionless mean velocity ($\mathbf{e}$ is a unit vector)

$$\mathcal{V}_i = \mathcal{V} e_i = \sqrt{\frac{m}{2 k_B T}} v_i. \tag{15}$$

The speed of sound is given by

$$c_s = \sqrt{\frac{\gamma k_B T}{m}}, \tag{16}$$

Here, $\gamma = 5/3$ for a monatomic gas. The Mach number is then

$$Ma = \sqrt{\frac{6}{5}} \mathcal{V}. \tag{17}$$

The distibution function is a Maxwellian, where $\mathcal{C}_i$ is replaced by $\mathcal{C}_i - \mathcal{V}_i$.
The particle influx is now calculated as

$$N_{(i)} = N_M [\exp(-\mathcal{V}_n^2) - \sqrt{\pi} \mathcal{V}_n (1 + \text{erf}(-\mathcal{V}_n))]. \tag{18}$$

If we restrict ourselves to terms up to second order in the Mach number, we can use

$$\text{erf}(x) \approx \frac{2}{\sqrt{\pi}} x,$$

giving

$$\frac{N_{(i)}}{N_M} = 1 + \mathcal{V}_n^2 - \sqrt{\pi} \mathcal{V}_n.$$

We find for the surface average (the term linear in $\mathcal{V}$ gives no contribution as the surface is closed)

$$(\overline{\frac{N_{(i)}}{N_M}}) = 1 + \mathcal{V}^2 N_{ij} e_i e_j. \tag{19}$$

Here, the purely geometric tensor $N_{ij}$ is ($S$ is the area of the body)

$$N_{ij} = \frac{1}{S}\int n_i n_j \mathrm{d}s. \tag{20}$$

It is convenient to split $N_{ij}$ into its irreducible components ($< ... >$ denotes symmetric traceless part)

$$N_{ij} = N_{<ij>} + \frac{1}{3}\delta_{ij}. \tag{21}$$

The eigenvalues of $N_{<ij>}$ lie in the interval $[-1/3, 2/3]$. Let us note that $N_{<ij>}$ vanishes for a sphere.
For an axially symmetric body, with axis $\mathbf{N}$, the tensor $N_{<ij>}$ is given by a scalar $a$,

$$N_{<ij>} = aN_{<i}N_{j>}. \tag{22}$$

Here,

$$a = \frac{3}{2}\frac{1}{S}\int [(\mathbf{n}\cdot\mathbf{N})^2 - \frac{1}{3}]\mathrm{d}s \tag{23}$$

For a flat body $a = 1$. For a needle shaped body $a = -1/2$.
Similarly we have

$$E_{(i)} = E_M\{(\frac{\mathcal{V}^2}{2}+1)\exp(-\mathcal{V}_n^2) - \sqrt{\pi}\mathcal{V}_n(\frac{\mathcal{V}^2}{2} + \frac{5}{4})[1 + \mathrm{erf}(-\mathcal{V}_n)]\}. \tag{24}$$

Approximately,

$$\frac{E_{(i)}}{E_M} = 1 + \frac{\mathcal{V}^2}{2} + \frac{3}{2}\mathcal{V}_n^2 - \frac{5}{4}\sqrt{\pi}\mathcal{V}_n.$$

The surface average is thus

$$\overline{(\frac{E_{(i)}}{E_M})} = 1 + \mathcal{V}^2(1 + \frac{3}{2}N_{<ij>}e_i e_j). \tag{25}$$

Inserting (19) and (25) into (13) we find the total influx of kinetic energy for a high conductivity body. (14) gives the equilibrium temperature to second order in the Mach number as

$$\frac{T_w}{T} = 1 + \frac{5Ma^2}{9}(1 + \frac{3}{4}N_{<ij>}e_i e_j). \tag{26}$$

## Axially symmetric body

Let us consider an axially symmetric body. If there is no external heat transfer to the body, in stationary conditions, it will have the temperature (with an error $\mathcal{V}^3$) - $\vartheta$ is the angle between the axis of the body and the direction of the flow -

$$\frac{T_w}{T} = 1 + \frac{5Ma^2}{9}[1 + \frac{a}{4}(3\cos^2\vartheta - 1)]. \tag{27}$$

For a sphere,

$$\frac{T_w}{T} = 1 + \frac{5}{9}Ma^2. \tag{28}$$

This is also the mean temperature if a small body of arbitrary shape is randomly oriented.
For a needle shaped body

$$\frac{T_w}{T} = 1 + \frac{5}{8}Ma^2(1 - \frac{1}{3}\cos^2\vartheta) \tag{29}$$

In particular, if the axis of the body is orthogonal to the flow,

$$\frac{T_w}{T} = 1 + \frac{5}{8}Ma^2. \tag{30}$$

For an infinite cylinder with axis orthogonal to the flow, this coincides with the result of Bell and Schaaf [4].

# SHEARING FLOW

## Chapman-Enskog solution

In a pure shearing flow, according to the Chapman-Enskog method, the distribution function is given by, see Chapman & Cowling [6]

$$f = f^{(0)}(1 + \phi).$$

$$\phi = -\frac{2}{n}B(\mathcal{C}^2)\mathcal{C}_{<i}\mathcal{C}_{j>}v_{<i,j>}.$$ (31)

$B$ is usually expanded in Sonine polynomials

$$B(\mathcal{C}^2) = \sum_{n=0}^{\infty} \tilde{b}_n S_{5/2}^{(n)}(\mathcal{C}^2).$$ (32)

The coefficient $\tilde{b}_0$ is related to the viscosity,

$$\mu = k_B T \tilde{b}_0$$

Let us introduce

$$\widehat{B}(\mathcal{C}^2) = B(\mathcal{C}^2)/\tilde{b}_0$$

Using the gas law $p = nk_BT$, we can write ($p_{<ij>} = -2\mu v_{<i,j>}$ is the viscous pressure tensor)

$$\phi = \frac{1}{p}\widehat{B}(\mathcal{C}^2)\mathcal{C}_{<i}\mathcal{C}_{j>}p_{<ij>}.$$ (33)

## Particle flux

The shearing gives the following contribution to the influx of particles

$$\frac{p_{<ij>}}{p}I_{ij}^N N_M.$$

Here, we have introduced the dimensionless integral

$$I_{ij}^N = -\frac{2}{\pi}\int_{\mathcal{C}_n<0} \mathcal{C}_n \widehat{B}(\mathcal{C}^2)\mathcal{C}_{<i}\mathcal{C}_{j>}e^{-\mathcal{C}^2}\mathrm{d}^3\mathcal{C} = I_N n_{<i}n_{j>}$$

The last expression follows from symmetry. Contracting twice with the normal vector and introducing spherical polar coordinates, we find, writing

$$\mathcal{C}_n = \mathcal{C}_k \cdot n_k,$$ (34)

$$I_N = -\frac{3}{\pi}\int_{\mathcal{C}_n<0} \mathcal{C}_n(\mathcal{C}_n^2 - \frac{1}{3}\mathcal{C}^2)\widehat{B}(\mathcal{C}^2)e^{-\mathcal{C}^2}\mathrm{d}^3\mathcal{C}$$

$$= -\frac{3}{\pi}\beta_N K.$$ (35)

Here,

$$\beta_N = \int_0^\infty \widehat{B}(\mathcal{C}^2)\mathcal{C}^5 e^{-\mathcal{C}^2}\mathrm{d}\mathcal{C}$$ (36)

$$= \frac{1}{2}\int_0^\infty \widehat{B}(x)x^2 e^{-x}\mathrm{d}x.$$

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Note that $\beta_N = 1$, if only the first term of the expansion of $\phi$ in Sonine polynomials is retained. This is usually a good approximation. - Further, the angular integral is

$$K = 2\pi \int_{\pi/2}^{\pi} (\cos^2 \theta - \frac{1}{3}) \sin \theta \cos \theta d\theta = -\frac{\pi}{6}.$$

We conclude that

$$I_N = \frac{1}{2} \beta_N.$$

The contribution from the shearing to the inflow rate of particles is thus

$$\frac{\beta_N}{2} \frac{p_{<ij>}}{p} n_{<i} n_{j>} N_M.$$

The total inflow of particles is then given by

$$N_{(i)} = N_M (1 + \frac{\beta_N}{2} \frac{p_{<ij>}}{p} n_{<i} n_{j>}) \tag{37}$$

Hence,

$$\overline{(\frac{N_{(i)}}{N_M})} = (1 + \frac{\beta_N}{2} \frac{p_{<ij>}}{p} N_{<ij>}) \tag{38}$$

## Flux of kinetic energy

Let us write the contribution from the shearing to $E_{(i)}$

$$-\frac{1}{2} \frac{p_{<ij>}}{p} I_{ij}^E E_M.$$

Here, the tensor $I_{ij}^E$ is (the expression to the right follows from symmetry)

$$I_{ij}^E = \frac{2}{\pi} \int_{\mathcal{C}_n < 0} \mathcal{C}^2 \mathcal{C}_n \widehat{B}(\mathcal{C}^2) \mathcal{C}_{<i} \mathcal{C}_{j>} e^{-\mathcal{C}^2} d^3 \mathcal{C}$$

$$= I_E n_{<i} n_{j>}.$$

Contracting $I_{ij}$ with $n_i n_j$ we find, using spherical polar coordinates,

$$I_E = \frac{3}{\pi} J_E K.$$

In this expression, the radial velocity integral is

$$J_E = \int_0^\infty \widehat{B}(\mathcal{C}^2) \mathcal{C}^7 e^{-\mathcal{C}^2} d\mathcal{C} = \frac{1}{2} \int_0^\infty \widehat{B}(x) x^3 e^{-x} dx.$$

Let us introduce a dimensionless coefficient

$$\beta_E = \frac{1}{6} \int_0^\infty \widehat{B}(x) x^3 e^{-x} dx. \tag{39}$$

Usually, the $n = 0$ term in (32) is the dominating one. If the other terms are neglected, $\beta_E$ takes the value 1 . - $J_E$ can now be written

$$J_E = 3\beta_E.$$

Hence,

$$I_E = -\frac{3}{2}\beta_E.$$

Thus the total rate at which heat is incoming is

$$E_{(i)} = (1 + \frac{3\beta_E}{4}\frac{p_{<ij>}}{p}n_{<i}n_{j>})E_M \tag{40}$$

We can easily estimate the size of the contribution from shearing. We have $\mu/p \sim \lambda/c_s$, where $\lambda$ is the mean free path and $c_s$ the speed of sound. If we estimate $v_{<i,j>} \sim v/l$, we find the relative size of the contribution to be $Ma\,Kn$, ($Ma = v/c_s$ the Mach number and $Kn = \lambda/l$ the Knudsen number.)

The total incident flux is thus

$$\overline{(\frac{E_{(i)}}{E_M})} = (1 + \frac{3\beta_E}{4}\frac{p_{<ij>}}{p}N_{<ij>}). \tag{41}$$

## Total rate of heat transferred to the body

### Body of arbitrary shape

The total influx of kinetic energy is now given by (13).

We see that also for $T_w = T$, there is a net influx given by

$$\frac{\beta}{4}\alpha_e\alpha_\tau E_M n_{<i}n_{j>}\frac{p_{<ij>}}{p}. \tag{42}$$

We have here introduced

$$\beta = 3\beta_E - 2\beta_N. \tag{43}$$

Note that $\beta = 1$, if only the first term in the Sonine polymial expansion of $\phi$ is retained.

Let us now consider a body, which adjusts its temperature so that the total inflow vanishes. To a first approximation in the shearing,

$$\frac{T_w}{T} = 1 + \frac{\beta}{4}\frac{p_{<ij>}}{p}N_{<ij>}, \tag{44}$$

where $\beta$ is given by (43).

### Axially symmetric bodies

Let us in particular consider an axially symmetric body in plane shear flow. In suitable coordinates, the only non-vanishing component of the velocity gradient is $v_{x,y}$. Let us denote the polar angles ($z$ is the polar axis) of the axial direction of the body by $\theta, \varphi$. Then

$$N_{<ij>}p_{<ij>} = \frac{a}{2}p_{xy}\sin^2\theta\sin(2\varphi).$$

In particular, for a body free to adjust its temperature until the total heat flow vanishes, the temperature of the body relaxes to

$$\frac{T_w}{T} = 1 + \frac{\beta a}{8}\frac{p_{xy}}{p}\sin^2\theta\sin(2\varphi). \tag{45}$$

For a needle-shaped body, putting $\beta = 1$ (neglecting higher Sonine polynomials), this coincides with the result of Bell and Schaaf [4].

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A needle-shaped body will then settle down to a temperature, which is higher than that of the surrounding gas, if the projection onto the plane of shearing of the axis of the body is in the first quadrant, but at a temperature, which is lower than that of the gas, if the axis is in the second quadrant.

Let us, finally, consider a body, which is kept at the temperature of the surrounding gas. We then have the total rate at which heat is transferred to the body

$$\frac{\beta a}{8}\alpha_e\alpha_\tau E_M S\frac{p_{xy}}{p}\sin^2\vartheta\sin(2\varphi). \tag{46}$$

We conclude that needle-shaped bodies absorb heat, when the projection of their axes on the plane of shearing lie in the first quadrant. They emit heat when their axes are in the second quadrant. For flat bodies the situation is reversed.

## TRANSLATION AND SHEARING

For an arbitrary flow field with shear, we can to the order considered add the two contributions to $\frac{T_w}{T}-1$ to find (we recall that $\mathbf{e}$ is the unit vector in the direction of the flow)

$$\frac{T_w}{T} = 1 + \frac{5}{9}Ma^2 + (\frac{5}{12}Ma^2 e_i e_j + \frac{\beta}{4}\frac{p_{<ij>}}{p})N_{<ij>}. \tag{47}$$

This is with an error $Ma^3, Kn\,Ma^2, Kn^2\,Ma$.

Let now also consider an axially symmetric body in a flow field, where the only non-vanishing velocity component $v_x$ is a function of $y$. This means that $\mathbf{e}\cdot\mathbf{N}=\cos\vartheta$ is replaced by $\sin\theta\cos\varphi$. We obtain

$$\frac{T_w}{T} = 1 + \frac{5Ma^2}{9}[1 + \frac{a}{4}(3\sin^2\theta\cos^2\varphi - 1)]$$

Totally,

$$\frac{T_w}{T} = 1 + \frac{5Ma^2}{9}(1 - \frac{a}{4}) + a\sin^2\theta[\frac{5Ma^2}{12}\cos^2\varphi + \frac{\beta}{8}\frac{p_{xy}}{p}\sin(2\varphi)]. \tag{48}$$

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## REFERENCES

1. Waldmann L.Z., *Z. Naturforschung* **A14**, 589 (1959).
2. Schaaf S.A., *Mechanics of Rarefied Gases. In Handbuch der Physik, VIII/2*, Berlin: Springer, 1963, p591.
3. Sone Y., *Journal de Mcanique Thorique et Applique* **3**, 316 (1984).
4. Bell S., and Schaaf S.A., *Jet Propulsion* **25**, 168 (1955).
5. Kogan, M.N., *Rarefied Gas Dynamics*, New York: Plenum, 1969.
6. Chapman S. and Cowling T.G., *The Mathematical Theory of Nonuniform Gases*, 3rd ed., Cambridge, England: Cambridge U.P., 1958.
7. Cercignani C., *The Boltzmann Equation and Its Applications*, New York: Springer-Verlag, 1988.
8. Bird G.A., *Molecular Gas Dynamics and the Direct Simulation of Gas Flows*, Oxford: Oxford U.P., 1994.